

ON BRAIDED NEAR-GROUP CATEGORIES

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ABSTRACT. We prove that any fusion category over \mathbb{C} with exactly one non-invertible simple object is spherical. Furthermore, we classify all such categories that come equipped with a braiding.

1. INTRODUCTION AND STATEMENTS OF RESULTS

This paper is devoted to the study of a type of fusion categories introduced by Jacob Siehler. For this paper, we will assume that the ground field is \mathbb{C} . Similar results can be found for any algebraically closed field of characteristic 0.

Definition 1.1. [Si1] *A near-group category is a semisimple, rigid tensor category with finitely many simple objects (up to isomorphism) such that all but one of the simple objects is invertible. In the language of fusion categories, a near-group category is a fusion category with one non-invertible simple object. If such a category comes equipped with a braiding, then we call it a braided near-group category.*

We show (see section 2) the well-known result that the Grothendieck ring of a near-group category is determined by a finite group G and a non-negative integer k . For each near-group category \mathcal{C} , we call the data (G, k) the near-group fusion rule of \mathcal{C} .

Example 1.2. (i) Near-group categories with fusion rule $(G, 0)$ for some finite group G are known as Tambara-Yamagami categories. These categories are classified up to tensor equivalence in [TY]. When they come equipped with a braiding, they are classified up to braided tensor equivalence in [Si1].

(ii) The well-known Yang Lee (see [O1]) categories are precisely the near-group categories with fusion rule $(1, 1)$. Up to tensor equivalence, there are two such categories, each of these admitting two braidings.

(iii) Let \mathcal{C} be the fusion category associated to the affine \mathfrak{sl}_2 on level 10 and let $A \in \mathcal{C}$ be the commutative \mathcal{C} -algebra of type E_6 . The category $\text{Rep}(A)$ of right A -modules contains a fusion subcategory (see [O2, Section 4.5]) which is a near-group category with fusion rule $(\mathbb{Z}/2\mathbb{Z}, 2)$.

(iv) The Izumi-Xu category \mathcal{IX} (see [CMS, Appendix A.4]) is a near-group category with fusion rule $(\mathbb{Z}/3\mathbb{Z}, 3)$.

J. Siehler has many results on near-group categories. With [Si2, Theorem 1.1] he proves that if $k \neq 0$, then $|G| \leq k + 1$. Siehler also classified braided near-group categories with near-group fusion rule $(G, 0)$ in [Si1, Theorem 1.2].

Let \mathcal{C} be a tensor category. Recall (see [DGNO]) that a spherical structure on \mathcal{C} is an isomorphism of tensor functors $\varphi : \text{Id} \rightarrow **$ so that for every simple object

$V \in \mathcal{C}$, we have

$$\dim(V) = \text{Tr}_V(\varphi) = \text{Tr}_{V^*}(\varphi) = \dim(V^*).$$

Let φ be an field automorphism of \mathbb{C} . Recall that a spherical structure is called φ -pseudounitary if $\varphi(\dim(V)) > 0$, for all simple objects V .

Our first main theorem, which is proved in section 2, is a result for (not necessarily braided) near-group categories. This result is positive evidence for the question [ENO] whether all fusion categories admit a spherical structure.

Theorem 1.3. *Any near-group category is spherical, moreover it is φ -pseudounitary for a suitable choice of φ .*

From the results of P. Deligne and G. Seitz, we are able to deduce the following classification of symmetric near-group categories.

Proposition 1.4. *Let \mathcal{C} be a symmetric near-group category with near-group fusion rule (G, k) and $k \neq 0$. Then \mathcal{C} is braided tensor equivalent to $\text{Rep}(\mathbb{F}_{p^l} \rtimes \mathbb{F}_{p^l}^*)$, for some $p^l \neq 2$.*

Our main result is from the study of non-symmetric braided near-group categories. We prove our main theorem in section 4.

Theorem 1.5. *Let \mathcal{C} be a non-symmetric, braided, near-group category with fusion rule (G, k) where $k \neq 0$, then G is either the trivial group, $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$. Furthermore if G is trivial, then there are four associated braided near group categories (up to braided tensor equivalence). All of these categories have fusion rule $(1, 1)$. If $G = \mathbb{Z}/2\mathbb{Z}$, then there are another two associated near-group categories, both with near-group fusion rule $(\mathbb{Z}/2\mathbb{Z}, 1)$. And finally, if $G = \mathbb{Z}/3\mathbb{Z}$, then there is one associated category with near-group fusion rule $(\mathbb{Z}/3\mathbb{Z}, 2)$.*

We note that Proposition 1.4, Theorem 1.5 and J. Siehler's classification of braided near-group categories with fusion rule $(G, 0)$ give a complete classification of braided near-group categories.

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2. NEAR-GROUP CATEGORIES ARE SPHERICAL

The goal of this section is to prove Theorem 1.3.

2.1. Near-group fusion rule. Let \mathcal{C} be a near-group category with non-invertible object X . The set of invertible objects of \mathcal{C} , denoted $\mathcal{O}(\mathcal{C})$, forms a group where multiplication is given by the tensor product structure on the category. Therefore we can associate a finite group G to any near-group category. Let $g \in G$ represent an invertible object of \mathcal{C} . Since g is invertible and X is not invertible, $g \otimes X$ is a non-invertible simple object of \mathcal{C} , therefore $g \otimes X \simeq X$ for all $g \in G$. Similarly, X^* is a non-invertible simple object, and therefore $X^* \simeq X$. Therefore

$$\text{Hom}(g, X \otimes X) \cong \text{Hom}(1, g^* \otimes X \otimes X) \cong \text{Hom}(1, X \otimes X) \neq 0.$$

Thus g appears as a summand of $X \otimes X$ for each $g \in G$. Since $\dim \operatorname{Hom}(1, X \otimes X) = 1$, g appears as a summand of $X \otimes X$ exactly once. Therefore we may decompose

$$X \otimes X \simeq \bigoplus_{g \in G} g \oplus kX$$

for some $k \in \mathbb{Z}_{\geq 0}$.

2.2. Sphericalization of a near-group category. For any fusion category, \mathcal{C} , we are given $\gamma : Id \rightarrow ***$ an isomorphism of tensor functors by [ENO, Theorem 2.6]. Then we may define the sphericalization of \mathcal{C} .

Definition 2.1. [ENO, Remark 3.1] *The sphericalization, $\tilde{\mathcal{C}}$, of a fusion category \mathcal{C} is the fusion category whose simple objects are pairs (V, α) where $V \in \mathcal{O}(\mathcal{C})$ and $\alpha : V \xrightarrow{\sim} V^{**}$ satisfies $\alpha^{**}\alpha = \gamma$. This category has a canonical spherical structure $i : Id \rightarrow ***$.*

Fix an isomorphism $f : V \rightarrow V^{**}$. Since $\operatorname{Hom}(V, V^{**})$ is one dimensional, we may write $\alpha = a \cdot f$ for some $a \in \mathbb{C}^\times$. We also have $\alpha^{**} = a \cdot f^{**}$. Similarly, we may write $\gamma = z \cdot f^{**}f$ for some $z \in \mathbb{C}^\times$. Then the condition $\alpha^{**}\alpha = \gamma$ is equivalent to $a^2 = z$. Therefore for each $V \in \mathcal{O}(\mathcal{C})$, we have two such α . Fixing one, we write $(V, \alpha) = V_+$ and $(V, -\alpha) = V_-$.

Now let \mathcal{C} be a near-group category with non-invertible simple object X and fusion rule (G, k) . For $X_+, X_- \in \tilde{\mathcal{C}}$, let $d = \dim(X_+) = \operatorname{Tr}_{X_+}(i) = \operatorname{Tr}_X(\alpha)$, so that $\dim(X_-) = \operatorname{Tr}_{X_-}(i) = \operatorname{Tr}_X(-\alpha) = -d$. Similarly for $g \in G$, define $g_+ \in \tilde{\mathcal{C}}$ to be the simple object with $\dim(g_+) = 1$ and whose image under the forgetful functor $\mathcal{F} : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ is g .

2.3. Some technical lemmas.

Lemma 2.2. *For $e \in G$ the identity, $e_- \otimes X_+ \simeq X_-$. Furthermore for $g, h \in G$, we have $g_+ \otimes X_\pm \simeq X_\pm$ and $g_+ \otimes h_+ \simeq (g \otimes h)_+$.*

Proof. Applying the forgetful functor $\mathcal{F}(e_- \otimes X_+) \simeq \mathcal{F}(e_-) \otimes \mathcal{F}(X_+) \simeq X$, and $\dim(e_- \otimes X_+) = -\dim(X_+)$, so $e_- \otimes X_+ \simeq X_-$.

Similarly we have $\mathcal{F}(g_+ \otimes X_+) \simeq \mathcal{F}(g_+) \otimes \mathcal{F}(X_+) \simeq g \otimes X \simeq X$, and $\dim(g_+ \otimes X_+) = \dim(g_+) \dim(X_+) = 1 \cdot d = d$. Therefore $g_+ \otimes X_+ \simeq X_+$. An analogous proof shows $g_+ \otimes X_- \simeq X_-$.

Finally $\mathcal{F}(g_+ \otimes h_+) \simeq g \otimes h$, and $\dim(g_+ \otimes h_+) = 1$, so $g_+ \otimes h_+ \simeq (g \otimes h)_+$. \square

Lemma 2.3. *Let \mathcal{C} be a near-group category with fusion rule (G, k) and non-invertible object X , and let $\tilde{\mathcal{C}}$ be the sphericalization of \mathcal{C} . We have $(X_\pm)^* \simeq X_\pm$, and furthermore*

$$X_+ \otimes X_+ \simeq X_- \otimes X_- \simeq \bigoplus_{g \in G} g_+ \oplus sX_+ \oplus tX_-,$$

where $s + t = k$.

Proof. Clearly the forgetful functor maps $(X_+)^* \mapsto X^*$, therefore $(X_+)^* \simeq X_+$ or X_- . Since $\dim(X_-) = -\dim(X_+)$, we conclude that $(X_+)^* \simeq X_+$.

Since $(X_+)^* \simeq X_+$, we have for each $g \in G$,

$$\operatorname{Hom}(g_+, X_+ \otimes X_+) \cong \operatorname{Hom}(1, (g^{-1})_+ \otimes X_+ \otimes X_+) \cong \operatorname{Hom}(1, X_+ \otimes X_+) \neq 0.$$

Therefore g_+ appears as a summand of $X_+ \otimes X_+$ for each $g \in G$. By applying the forgetful functor, we see that g_+ appears as a summand at most once. This gives us

$$X_+ \otimes X_+ \simeq \bigoplus_{g \in G} g_+ \oplus sX_+ \oplus tX_-,$$

with no restriction on s, t . Again applying the forgetful functor gives

$$X \otimes X \simeq \bigoplus_{g \in G} g \oplus (s+t)X,$$

and the lemma is proved after noting

$$X_- \otimes X_- \simeq (e_- \otimes X_+) \otimes (e_- \otimes X_+) \simeq X_+ \otimes X_+.$$

□

After renaming of X_+ , we may assume $s - t \geq 0$.

Lemma 2.4. *For $X_+ \in \tilde{C}$, we have $d = \dim(X_+) = \frac{r \pm \sqrt{r^2 + 4n}}{2}$ and $\dim(\mathcal{C}) = \frac{r^2 + 4n \pm r\sqrt{r^2 + 4n}}{2}$ where $n = |G|$ and $r = s - t \geq 0$.*

Proof. We have

$$d^2 = \dim(X_+ \otimes X_+) = \dim \left(\bigoplus_{g \in G} \oplus sX_+ \oplus tX_- \right) = |G| + (s - t)d = n + rd.$$

And

$$\dim(\mathcal{C}) = |G| + d^2 = 2n + rd = \frac{r^2 + 4n \pm r\sqrt{r^2 + 4n}}{2}.$$

□

Lemma 2.5. *Recall $r = s - t$.*

- (a) *If \tilde{C} is pseudo-unitary, then $r = k$,*
- (b) *If $\sqrt{r^2 + 4n} \in \mathbb{Z}$, then $r = k$,*
- (c) *If $\sqrt{k^2 + 4n} \in \mathbb{Z}$, then $r = k$.*

Proof. (a) If \tilde{C} is pseudo-unitary, then $\dim(\tilde{C}) = \text{FPdim}(\tilde{C}) = 2\text{FPdim}(\mathcal{C})$. Therefore $r^2 + 4n + r\sqrt{r^2 + 4n} = k^2 + 4n + k\sqrt{k^2 + 4n}$, and since $|r| = |s - t| \leq k$, we have $r = k$.

(b) If $\sqrt{r^2 + 4n} \in \mathbb{Z}$, then d is a rational algebraic integer, therefore $d \in \mathbb{Z}$. By [HR, Lemma A.1] \tilde{C} is pseudo-unitary, and $r = k$ by (a).

(c) If $\sqrt{k^2 + 4n} \in \mathbb{Z}$, then $\text{FPdim}(X) \in \mathbb{Z}$. By [ENO, Proposition 8.24] \tilde{C} is pseudo-unitary and $r = k$ by (a).

□

We will also use the following well-known lemma about algebraic integers.

Lemma 2.6. *Let $a, b, c, d \in \mathbb{Z}$ such that $\sqrt{b}, \sqrt{d} \notin \mathbb{Z}$. Then $\frac{a + \sqrt{b}}{c + \sqrt{d}}$ is an algebraic integer if and only if $\frac{a - \sqrt{b}}{c - \sqrt{d}}$ is an algebraic integer.*

Proof. Since $b \in \mathbb{Z}$ is not a square, we may write $b = m \cdot p_1^{\beta_1} \cdots p_k^{\beta_k}$ for some square $m \in \mathbb{Z}$ and primes p_1, \dots, p_k and odd integers β_1, \dots, β_k . Similarly, we may write $d = n \cdot q_1^{\delta_1} \cdots q_l^{\delta_l}$ for some square n , primes q_1, \dots, q_l and odd integers $\delta_1, \dots, \delta_l$. We will consider two cases.

Case (i): Up to ordering $p_1 = q_1$. Let $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{b}, \sqrt{d})/\mathbb{Q})$ be the element which maps $\sqrt{p_1}$ to $-\sqrt{p_1}$ and fixes $\sqrt{p_i}$ for $i \neq 1$ and $\sqrt{q_j}$ for $j \neq 1$. Then $\sigma(a + \sqrt{b}) = a - \sqrt{b}$, $\sigma(c + \sqrt{d}) = c - \sqrt{d}$ and $\sigma\left(\frac{a+\sqrt{b}}{c+\sqrt{d}}\right) = \frac{a-\sqrt{b}}{c-\sqrt{d}}$.

Case (ii): $p_i \neq q_j$ for all $1 \leq i \leq k, 1 \leq j \leq l$. Then let $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{b}, \sqrt{d})/BQ)$ be the element which maps $\sqrt{p_1}$ to $-\sqrt{p_1}$, maps $\sqrt{q_1}$ to $-\sqrt{q_1}$ and fixes $\sqrt{p_i}$ for $i \neq 1$ and $\sqrt{q_j}$ for $j \neq 1$. Then as above $\sigma\left(\frac{a+\sqrt{b}}{c+\sqrt{d}}\right) = \frac{a-\sqrt{b}}{c-\sqrt{d}}$. \square

2.4. Proof of Theorem 1.3. Let $D = \dim(\tilde{\mathcal{C}}) = r^2 + 4n + r\sqrt{r^2 + 4n}$, and $\Delta = \text{FPdim}(\tilde{\mathcal{C}}) = k^2 + 4n + k\sqrt{k^2 + 4n}$. Then by [ENO, Proposition 8.22]

$$\frac{D}{\Delta} = \frac{r^2 + 4n + r\sqrt{r^2 + 4n}}{k^2 + 4n + k\sqrt{k^2 + 4n}}$$

is an algebraic integer. Our goal is to prove that $r = k$, thus proving the theorem. When either $\sqrt{r^2 + 4n}$ or $\sqrt{k^2 + 4n}$ are integers, we know $r = k$ by Lemma 2.5. Therefore assume $\sqrt{r^2 + 4n}, \sqrt{k^2 + 4n} \notin \mathbb{Z}$. Then

$$\frac{r^2 + 4n - r\sqrt{r^2 + 4n}}{k^2 + 4n - k\sqrt{k^2 + 4n}}$$

is an algebraic integer by Lemma 2.6, and thus

$$\left(\frac{r^2 + 4n + r\sqrt{r^2 + 4n}}{k^2 + 4n + k\sqrt{k^2 + 4n}}\right) \left(\frac{r^2 + 4n - r\sqrt{r^2 + 4n}}{k^2 + 4n - k\sqrt{k^2 + 4n}}\right) = \frac{4n(r^2 + 4n)}{4n(k^2 + 4n)} = \frac{r^2 + 4n}{k^2 + 4n}$$

is an algebraic integer. Therefore $r^2 = k^2$, and $r = k$, since $r \geq 0$.

The full tensor category generated by simple objects $\{g_+\}_{g \in G} \cup \{X_+\}$ is tensor equivalent (by the forgetful functor) to \mathcal{C} . Therefore \mathcal{C} is tensor equivalent to a full tensor subcategory of a spherical category and therefore spherical itself. Moreover if $d > 0$, then \mathcal{C} is pseudo-unitary, and if $d < 0$, then \mathcal{C} is φ -pseudounitary. \square

2.5. Near-group categories with integer Frobenius-Perron dimension.

Proposition 2.7. *If a near-group category \mathcal{C} with near-group fusion rule (G, k) has integer Frobenius-Perron dimension, then either $k = 0$ or $k = |G| - 1$. In the latter case $\text{FPdim}(\mathcal{C}) = |G|(|G| + 1)$.*

Proof. $\text{FPdim}(X) = \frac{1}{2}(k + \sqrt{k^2 + 4n})$. Therefore if $\text{FPdim}(\mathcal{C}) \in \mathbb{Z}$, then $\text{FPdim}(X)^2 = \frac{1}{2}(k^2 + 2n + k\sqrt{k^2 + 4n})$ is an integer, and $\sqrt{k^2 + 4n} \in \mathbb{Z}$. Therefore $k^2 + 4n = (k + l)^2$ for some $l \in \mathbb{Z}_{>0}$. Expanding, we get $4n = 2kl + l^2$. Therefore l is even and $l = 2p$ for $p \in \mathbb{Z}_{>0}$. Finally, $k + 1 \leq kp + p^2 \leq n \leq k + 1$ by [Si2, Theorem 1.1] when $k \neq 0$.

Therefore $k = 0$ or $k = n - 1 = |G| - 1$. In the latter case, $\text{FPdim}(X) = k + 1 = |G|$, and $\text{FPdim}(\mathcal{C}) = n + (k + 1)^2 = n + n^2 = |G|(|G| + 1)$. \square

3. MÜGER CENTER OF A BRAIDED NEAR-GROUP CATEGORY

The goal of this section is to prove Proposition 1.4. We will use mostly definitions and results from [DGNO].

Let \mathcal{C} be a braided tensor category. From [Mu2], we define the Müger center of \mathcal{C} to be the full tensor subcategory of \mathcal{C} with objects

$$\{X \in \mathcal{C} \mid \sigma_{Y,X} \sigma_{X,Y} = \text{id}_{X \otimes Y} \ \forall Y \in \mathcal{C}\}.$$

We denote the Müger center of \mathcal{C} by \mathcal{C}' .

3.1. The Müger center of a near-group category contains all invertible objects. Recall the following definitions for braided fusion categories.

Definition 3.1. [DGNO, Section 2.2; Section 3.3] *Let \mathcal{C} be a fusion category:*

- (a) *Define \mathcal{C}_{ad} to be the fusion subcategory generated by $Y \otimes Y^*$ for $Y \in \mathcal{O}(\mathcal{C})$.*
- (b) *For \mathcal{K} a fusion subcategory of \mathcal{C} , we define the commutator of \mathcal{K} to be the fusion subcategory $\mathcal{K}^{co} \subseteq \mathcal{C}$, generated by all simple objects $Y \in \mathcal{O}(\mathcal{C})$, where $Y \otimes Y^* \in \mathcal{O}(\mathcal{K})$.*

Let \mathcal{C} be a near-group category with near-group fusion rule (G, k) . Recall that for this proposition we assume $k \neq 0$.

Lemma 3.2. $\mathcal{C}_{ad} = \mathcal{C}$.

Proof. This is clear as $X \simeq X^*$ and $X \otimes X \simeq G \oplus kX$, thus contains all simple objects of \mathcal{C} as summands. \square

Lemma 3.3. [DGNO, Proposition 3.25] *Let \mathcal{K} be a fusion subcategory of a braided fusion category \mathcal{C} . Then $(\mathcal{K}_{ad})' = (\mathcal{K}')^{co}$.*

Letting $\mathcal{K} = \mathcal{C}$ in Lemma 3.3, we get

Proposition 3.4. *Let \mathcal{C} be a braided near-group category with fusion rule (G, k) . If $k > 0$, then the Müger center $\mathcal{C}' \subseteq \mathcal{C}$ is either \mathcal{C} or Vec_G .*

Proof. $\mathcal{C}' = (\mathcal{C}_{ad})' = (\mathcal{C}')^{co} \supset G$. \square

In particular, \mathcal{C} can only be modular if G is trivial.

3.2. Symmetric tensor categories. Let A be a group. Deligne [D] defines $\text{Rep}(A, z)$ to be the category of finite dimensional super representations (V, ρ) of A , where $\rho(z)$ is the automorphism of parity of V . In [DGNO] this is presented as the fusion category $\text{Rep}(G)$ with $z \in Z(G)$ satisfying $z^2 = 1$ and braiding σ' given by

$$\sigma'_{UV}(u \otimes v) = (-1)^{ij} v \otimes u \text{ if } u \in U, v \in V, zu = (-1)^i u, zv = (-1)^j v.$$

In [D, Corollaire 0.8] it is shown that any symmetric fusion category is equivalent to $\text{Rep}(A, z)$ for some choice of finite group A , and central element $z \in A$ with $z^2 = 1$. If $z \neq 1$ we call such a category super-Tannakian. If $z = 1$, then $\text{Rep}(A, z) = \text{Rep}(A)$ and it is called Tannakian. Note that $\text{Rep}(A/\langle z \rangle)$ is the subcategory of modules M where z acts trivially on M . This is a maximal Tannakian subcategory of $\text{Rep}(A, z)$.

Recall (see [DGNO, Example 2.42]) $s\text{Vec}$ is defined to be the category $\text{Rep}(\mathbb{Z}/2\mathbb{Z}, z)$, where z is the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$. The following lemma is due to [Mu1, Lemma 5.4] and [DGNO, Lemma 3.28]. This lemma will be used to show that particular categories do not exist.

Lemma 3.5. *Let \mathcal{C} be a braided fusion category and $\delta \in \mathcal{C}'$ an invertible object such that the fusion subcategory of \mathcal{C} generated by δ is braided equivalent to $s\text{Vec}$. Then for all $V \in \mathcal{O}(\mathcal{C})$, $\delta \otimes V$ cannot be mapped to V by some tensor automorphism.*

Proof. For $V \in \mathcal{O}(\mathcal{C})$, let μ_V be defined to be the composition

$$V \xrightarrow{\text{id}_V \otimes \text{coeval}_{V^*}} V \otimes V^* \otimes V^{**} \xrightarrow{\sigma_{V,V^*} \otimes \text{id}_{V^{**}}} V^* \otimes V \otimes V^{**} \xrightarrow{\text{eval}_V \otimes \text{id}_{V^{**}}} V^{**}$$

where σ is the braiding on \mathcal{C} . It is well known (see [BK, Lemma 2.2.2]) that for $V, W \in \mathcal{O}(\mathcal{C})$, μ_V and μ_W satisfy

$$\mu_V \otimes \mu_W = \mu_{V \otimes W} \sigma_{W,V} \sigma_{V,W}.$$

Therefore since, $\delta \in \mathcal{C}'$, we have

$$\mu_{\delta \otimes V} = \mu_\delta \otimes \mu_V,$$

for all $V \in \mathcal{O}(\mathcal{C})$. Since δ generates $s\text{Vec}$, we know that $\sigma'(\delta, \delta) = -\text{id}_1$ and $\mu_\delta = -\text{id}_\delta$. Recall [ENO] for a simple object $U \in \mathcal{O}(\mathcal{C})$, we define $d_+(U)$ to be the composition

$$1 \xrightarrow{\text{coeval}_V} V \otimes V^* \xrightarrow{\mu \otimes \text{id}_{V^*}} X^{**} \otimes V^* \xrightarrow{\text{eval}_{V^*}} 1.$$

Then

$$\begin{aligned} d_+(\delta \otimes V) &= \text{eval}_{(\delta \otimes V)^*} \circ (\mu_{\delta \otimes V} \otimes \text{id}_1) \circ \text{coeval}_{\delta \otimes V} \\ &= \text{eval}_{\delta^*} \circ (-\text{id}_\delta \otimes \text{id}_1) \circ \text{coeval}_\delta \cdot \text{eval}_{V^*} \circ (\mu_V \otimes \text{id}_1) \text{coeval}_V \\ &= -d_+(V). \end{aligned}$$

Since $d_+(V) \neq 0$, we have $d_+(V) \neq d_+(\delta \otimes V)$, and V cannot be mapped to $\delta \otimes V$ by some automorphism. \square

3.3. Proof of Proposition 1.4. Let \mathcal{C} be a symmetric near-group category with near-group fusion rule (G, k) . by [D, Corollaire 0.8] \mathcal{C} is equivalent (as a tensor category) to $\text{Rep}(H)$ for some finite group H . Since \mathcal{C} is a near-group category, H has exactly one irreducible representation of dimension greater than one. The following lemma classifies such groups.

Lemma 3.6. [Se] *A group G has exactly one irreducible \mathcal{C} -representation of degree greater than one if and only if (i) $|G| = 2^k$, k is odd, $[G, G] = Z(G)$, and $|[G, G]| = 2$, or (ii) G is isomorphic to the group of all transformations $x \mapsto ax + b$, $a \neq 0$, on a field of order $p^n \neq 2$.*

By [D, Corollaire 0.8] and Lemma 3.6, \mathcal{C} is tensor equivalent to $\text{Rep}(H)$ where $|H| = 2^l$, or H is isomorphic to the group of all transformations $x \mapsto ax + b$, $a \neq 0$, on a field of order $p^l \neq 2$.

If $|G| = 2^l$, then by Lemma 2.7, $\text{Rep}(G)$ is Tambara-Yamagami if it is near-group. Therefore we may assume that H is the latter group described above. Such a group H is isomorphic to $\mathbb{F}_{p^l} \rtimes \mathbb{F}_{p^l}^*$ since there is a split short exact sequence

$$1 \rightarrow \mathbb{F}_{p^l} \rightarrow H \rightarrow \mathbb{F}_{p^l}^* \rightarrow 1.$$

Therefore \mathcal{C} is tensor equivalent to $\text{Rep}(\mathbb{F}_{p^l} \rtimes \mathbb{F}_{p^l}^*)$. Since $Z(H) = 1$, there does not exist a braiding on $\text{Rep}(H)$ making it a super-Tannakian category. Therefore \mathcal{C} is braided tensor equivalent to $\text{Rep}(\mathbb{F}_{p^l} \rtimes \mathbb{F}_{p^l}^*)$. \square

3.4. Equivariantization of a braided tensor category. Let \mathcal{C} be a braided tensor category.

Definition 3.7. [DGNO, Section 4.2] (i) Let $\underline{\text{Aut}}^{\text{br}}(\mathcal{C})$ be the category whose objects are braided tensor equivalences and whose morphisms are isomorphism of braided tensor functors.

(ii) For a group G , let \underline{G} be the tensor category whose objects are elements of G , whose morphisms are the identity morphisms and whose tensor product is given by group multiplication.

(iii) We say that G acts on \mathcal{C} viewed as a braided tensor category if there is a monoidal functor $\underline{G} \rightarrow \underline{\text{Aut}}^{\text{br}}(\mathcal{C})$.

(iv) We say \mathcal{C} is a braided tensor category \mathcal{C} over \mathcal{E} if it is equipped with a braided functor $\mathcal{E} \rightarrow \mathcal{C}'$.

Let G be a group, and G act on \mathcal{C} viewed as a braided tensor category. Then define the equivariantization of \mathcal{C} by G .

Definition 3.8. [DGNO, Definition 4.2.2] Let \mathcal{C}^G be the category with objects G -equivariant objects. That is an object $X \in \mathcal{C}$ along with an isomorphism $\mu_g : F_g(X) \rightarrow X$ such that the following diagram commutes for all $g, h \in G$.

$$\begin{array}{ccc} F_g(F_h(X)) & \xrightarrow{F_g(u_h)} & F_g(X) \\ \gamma_{g,h}(X) \downarrow & & \downarrow u_g \\ F_{gh}(X) & \xrightarrow{u_{gh}} & X \end{array}$$

The morphisms in \mathcal{C}^G are morphisms in \mathcal{C} which commute with u_g . The tensor product on \mathcal{C}^G is the obvious one induced by the tensor product on \mathcal{C} . Since the action of G on \mathcal{C} respects the braiding, there is an induced braiding on \mathcal{C}^G .

The following proposition from [DGNO] relates actions of G on \mathcal{C} and equivariantization.

Proposition 3.9. [DGNO, Theorem 4.18(ii)] Let G be a finite group and \mathcal{C} be a braided tensor category over $\text{Rep}(G)$. Then there is a braided tensor category \mathcal{D} equipped with an action of G , such that $\mathcal{D}^G \cong \mathcal{C}$.

3.5. Tannakian centers of braided near-group categories. Let \mathcal{C} be a braided near-group category with near-group fusion rule (G, k) . Assume that \mathcal{C} is not symmetric, so $\mathcal{C}' = \text{Vec}_G$. Therefore $\mathcal{C}' = \text{Rep}(A, z)$ for some choice of finite group A and $z \in A$. For the remainder of this section, we will assume that $z \neq 1$ and derive a contradiction.

Recall $H := \text{Rep}(A/\langle z \rangle) \subseteq \mathcal{C}'$ is a maximal Tannakian subcategory of $\text{Rep}(A, z)$. By Proposition 3.9, there exists a braided fusion category \mathcal{D} and an action of H on \mathcal{D} so that $\mathcal{D}^H = \mathcal{C}$ and $\text{Vec}^H = \text{Rep}(H)$. Since \mathcal{C}' is a braided tensor category over $\text{Rep}(H)$, there also exists a category $\mathcal{D}_1 \subset \mathcal{D}$ such that $\mathcal{D}_1^H = \mathcal{C}'$. By [DGNO, Proposition 4.26], we have $\text{FPdim}(\mathcal{D}_1) = \text{FPdim}(\mathcal{C}')/|H| = 2$. Let $\mathcal{O}(\mathcal{D}_1) = \{1, Z\}$, and $\mathcal{O}_1, \dots, \mathcal{O}_m$ denote the orbits of the simple objects of \mathcal{D} under the action of H , where $\mathcal{O}_1 = \{1\}$ and $\mathcal{O}_2 = \{Z\}$. Since $\mathcal{D}_1^H = \mathcal{C}'$ and there is only one simple object of \mathcal{C} not contained in \mathcal{C}' , there are only three orbits. The following proposition shows that no such category can exist.

Proposition 3.10. *There are no super-Tannakian categories \mathcal{T} with the following structure:*

- (i) $\mathcal{O}(\mathcal{T}) = \{1, \delta, T_1, \dots, T_q\}$ where the fusion subcategory of \mathcal{T} generated by δ is braided equivalent to $s\text{Vec}$,
- (ii) An action of a group A on \mathcal{T} transitively permuting $\{T_1, \dots, T_q\}$.

Proof. Since δ is invertible, $\delta \otimes T_1 \simeq T_s$ for some $1 \leq s \leq q$. By Lemma 3.5 there is no automorphism mapping $\delta \otimes T_1$ to T_s . This contradicts the assumption that A acts transitively on $\{T_1, \dots, T_q\}$. \square

Therefore we proved the following proposition.

Proposition 3.11. *If \mathcal{C} is a non-symmetric, braided, near-group category and $k \neq 0$, then the Müger center $\mathcal{C}' = \text{Rep}(H)$ for some abelian group H .*

4. CLASSIFICATION OF NON-SYMMETRIC BRAIDED NEAR-GROUP CATEGORIES.

The goal of this section is to show there are 7 non-symmetric, braided, near-group categories (up to braided tensor equivalence) which are not Tambara-Yamagami. Again, we only care about the case when $k \neq 0$, as J. Siehler already did the classification when $k = 0$ [Si1].

In the previous section, we proved $\mathcal{C}' = \text{Rep}(H)$. By Theorem 3.9, there exists a braided fusion category \mathcal{D} and an action of H on \mathcal{D} so that $\mathcal{D}^H = \mathcal{C}$ and $\text{Vec}^H = G$. Let $\mathcal{O}_1, \dots, \mathcal{O}_m$ denote the orbits of the simple objects of \mathcal{D} under the action of H , where $\mathcal{O}_1 = \{1\}$. Since $\text{Vec}^H = G$, we have $m = 2$, otherwise $G \cup \{X\} \subsetneq \mathcal{O}(\mathcal{C})$. For the remainder of this section, let $\mathcal{O}_2 = \{D_1, \dots, D_s\}$.

Lemma 4.1. *If $s > 1$, then $\dim(D_j) = 1$ for $1 \leq j \leq s$.*

Proof. If $s \geq 2$, then there exists $1 \leq i \leq s$, such that $D_i^* \not\cong D_1$. Therefore $D_1 \otimes D_i = \bigoplus_{j=1}^s a_j D_j$, and $d = \dim(D_j)$ satisfies the identity $d^2 = d \sum_{j=1}^s a_j$. This gives $d \in \mathbb{Z}$, so by the proof of [ENO, Proposition 8.22], d divides $\text{FPdim}(\mathcal{D})$. Therefore, since $\text{FPdim}(\mathcal{D}) = 1 + sd^2$, we have $d = 1$. \square

Lemma 4.2. *If $\mathcal{O}_2 = \{D_1, \dots, D_s\}$, then s is either 1 or n .*

Proof. By Lemma 4.1, if $s \geq 2$, then $\text{FPdim}(\mathcal{D}) = 1 + s$. Therefore $\text{FPdim}(\mathcal{C}) = n(1 + s) \in \mathbb{Z}$ and $1 + s = \text{FPdim}(\mathcal{D}) = \frac{1}{n} \text{FPdim}(\mathcal{C}) = n + 1$, since $\text{FPdim}(\mathcal{C}) = n(n + 1)$ by Proposition 2.7. \square

Proposition 4.3. *If $\mathcal{O}(\mathcal{D}) = \{1, D_1\}$, then either $\mathcal{C} = \mathcal{D}^H$ is a Yang-Lee category or \mathcal{C} is Tambara-Yamagami. Moreover there are (up to braided equivalence) four braided near-group categories \mathcal{C} which are not Tambara-Yamagami and \mathcal{C}_H is of rank two.*

Proof. Let $D = D_1$. Assume $D \otimes D = 1$. Let X be the non-invertible object of \mathcal{C} . Therefore X is an equivariant object under the action of H on \mathcal{C} . Therefore $X = mD$ for some integer m . Therefore $X \otimes X = m^2 1$ in \mathcal{C} and must therefore lie in $\text{Rep}(H)$ in \mathcal{C} . In this case \mathcal{C} is Tambara-Yamagami.

Now assume $D \otimes D = 1 \oplus D$. Therefore H acts on \mathcal{D} trivially. This gives $\mathcal{C} = \mathcal{D} \boxtimes \text{Rep}(H)$, which is only a near-group category when H is trivial and $\mathcal{C} = \mathcal{D}$.

The last part of the proposition is simply a note that there are four Yang-Lee categories up to braided equivalence [O1]. \square

Since we just classified the case when \mathcal{D} is of rank two, we will assume for the remainder of this section that $s > 1$, and therefore by Lemma 4.1, \mathcal{D} is a pointed braided category which is non-degenerate by [DGNO, Corollary 4.30] since $\mathcal{D}^H = \mathcal{C}$, where $\mathcal{C}' = \text{Rep}(H)$. It is shown (see [DGNO] or [JS]) that a non-degenerate pointed braided category is classified by an abelian group A and a non-degenerate quadratic form $q : A \rightarrow \mathbb{C}^\times$ on A . Note that A is the group of isomorphism classes of simple objects. They denote such a category by $\mathcal{C}(A, q)$. Recall that the data (A, q) for a finite abelian group A and a non-degenerate quadratic form $q : A \rightarrow \mathbb{C}^\times$ is called a metric group.

Proposition 4.4. *If \mathcal{D} is of rank at least three, then A is either $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$. Moreover if:*

- (i) $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then $q(a) = -1$ for every non-trivial element of A ,
- (ii) $A = \mathbb{Z}/3\mathbb{Z}$, then $q(a) = q(b)$ are primitive third roots of unity for both non-trivial elements a, b of A .

Proof. $\mathcal{O}(\mathcal{D}) = \{1, D_1, \dots, D_p\}$ where H acts transitively on $\{D_1, \dots, D_s\}$ by braided-tensor functors. Let $A = \{e, d_1, \dots, d_s\}$, then $o(d_i) = o(d_j)$ for $1 \leq i, j \leq s$. Therefore A is an elementary abelian group. Let $p = o(d_1)$. The action of H on \mathcal{D} gives rise to an action of H on the metric group (A, q) by morphisms $\{\varphi_h\}_{h \in H}$ of metric groups. Since H acts transitively on \mathcal{O}_2 , we have for any $1 \leq i, j \leq s$ we have $h \in H$ so that $\varphi_h(d_i) = d_j$. Since φ_h is a morphism of metric groups, we have $q(d_j) = q(\varphi_h(d_i)) = q(d_i)$. Therefore it makes sense to define $\omega = q(d_i)$. By [DGNO, Remark 2.37 (i)], we have $1 = q(e) = q(d_1^p) = \omega^{p^2}$, therefore ω is a root of unity.

[DGNO, Corollary 6.3] states that for (A, q) a metric group, we have

$$\left| \sum_{a \in A} q(a) \right|^2 = |A|.$$

Therefore if $|A| = m$, we have

$$m = |1 + (m-1)\omega|^2 \geq (|(m-1)\omega| - 1)^2 = (m-2)^2 = m^2 - 4m + 4.$$

This gives $(m-1)(m-4) \leq 0$, so $m = 2, 3$ or 4 and we assume $m \geq 3$. Since A is an elementary abelian group, we know that $m = 4$ implies $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Assume $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then $|1 + 3\omega|^2 = 4$, giving $\omega = -1$.

Finally, for $A = \mathbb{Z}/3\mathbb{Z}$, we have $\omega = q(d_1) = q(d_1^2) = \omega^4$, so ω is a third root of unity. This gives $|1 + 2\omega|^2 = 3$, and ω is a primitive third root of unity. \square

Proposition 4.5. *Let \mathcal{C} be a non-symmetric braided near group category with fusion rule (G, k) where $k \neq 0$. There are two such categories (up to braided tensor equivalence) when $G = \mathbb{Z}/2\mathbb{Z}$ and one such category (up to braided tensor equivalence) when $G = \mathbb{Z}/3\mathbb{Z}$.*

Proof. Assume $G = \mathbb{Z}/2\mathbb{Z}$. We have shown above that $\mathcal{C} = \mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)^H$, where $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}} = \text{Rep}(H)$ (therefore $H = \mathbb{Z}/2\mathbb{Z}$) and $q : \mathbb{Z}/3\mathbb{Z} \rightarrow \mathbb{C}^\times$ is defined by $q(a) = q(b)$ is a primitive third root of unity for non-trivial elements $a, b \in \mathbb{Z}/3\mathbb{Z}$. For each of the two choices of q , we have one non-trivial action of H on $\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)$. Therefore there are two non-symmetric near-group categories with fusion rule $(\mathbb{Z}/2\mathbb{Z}, 1)$.

Assume $G = \mathbb{Z}/3\mathbb{Z}$. Then we showed that $\mathcal{C} = \mathcal{C}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, q)^H$, where $\text{Vec}_{\mathbb{Z}/3\mathbb{Z}} = \text{Rep}(H)$ (therefore $H = \mathbb{Z}/3\mathbb{Z}$) and $q : \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{C}^\times$ is defined by

$q(a) = q(b) = -1$ for the generators of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Again, we only have one non-trivial action of $\mathbb{Z}/3\mathbb{Z}$ on $\mathcal{C}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, q)$. Therefore we get one non-symmetric near-group category with fusion rule $(\mathbb{Z}/3\mathbb{Z}, 2)$. \square

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